Massless Particles in QFT from Algebras without Involution

Glenn Eric Johnson

Oak Hill, VA.[*]

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Abstract: The explicit realizations of quantum field theory (QFT) admitted by a revision to the Wightman axioms for the vacuum expectation values (VEV) of fields includes massless particles when there are four or more spacetime dimensions.

1 Introduction

It was demonstrated in [1] that explicit realizations of quantum field theory (QFT) are admitted by a revision to the Wightman axioms for the vacuum expectation values (VEV) of fields. The expansion of QFT beyond consideration of ∗-algebras achieved realizations of the first principles of quantum mechanics, Poincaré covariance, positive energy, and microcausality for fields exhibiting interaction. Here the development is extended to include massless particles.

Massless particles require a different algebra of function sequences and four spacetime dimensions to demonstrate that the VEV are continuous linear functionals. The demonstrations of: a semi-norm for a subalgebra $\mathcal{B}$ within an enveloping ∗-algebra of function sequences $\mathcal{A}$; Poincaré covariance; locality; spectral support; and a unique vacuum (indecomposibility) do not depend on whether the constituent particles have finite mass and the original demonstrations [1] suffice. In the case of massless particles, evaluation of scattering amplitudes succeeds just as in the finite mass case although the LSZ asymptotic states, as defined here, are no longer test functions on mass shells. The revised Wightman axioms remain satisfied with the inclusion of massless particles.

1.1 Definitions and notation

This note employs the same notation as [1]. Spacetime coordinates x and energy-momentum vectors p are $x := t, \mathbf{x}$, $p := E, \mathbf{p}$ and more generally $q := q_{(0)}, \mathbf{q}$ with $x^2 := x^T g x = t^2 - \mathbf{x}^2$ for the Minkowski metric g. $x, p, q \in \mathbf{R}^d$, $\mathbf{x}, \mathbf{p}, \mathbf{q} \in \mathbf{R}^{d-1}$, $\mathbf{x}^2 := x_{(1)}^2 + \ldots x_{(d-1)}^2$ is the square of the Euclidean length in $\mathbf{R}^{d-1}$, and $E_j^2 = \omega_j^2 := m_{\kappa_j}^2 + \mathbf{p}_j^2$ describe mass shells. $p \in \bar{V}^+$, the closed forward cone, if $p^2 \geq 0$ and $E \geq 0$. Multiple variables are denoted by $(x)_n := x_1, x_2 \ldots x_n$ and $(x)_{k,n} := x_k, \ldots x_n$ for either ascending or descending sequences of indices. Repetition of variables includes recursion, for example,

$$(\sum_\nu (\int d\zeta)_2)_3 := \sum_{\nu_1} \int d\zeta_1 \int d\zeta_2 \sum_{\nu_2} \int d\zeta_3 \int d\zeta_4 \sum_{\nu_3} \int d\zeta_5 \int d\zeta_6. \tag{1}$$

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[*]Author is reached at: glenn.e.johnson@gmail.com.

Dirac delta generalized functions supported on mass shells are denoted

$$\begin{aligned} \delta_j^{\pm} &:= \delta(\pm E_j - \omega_j)/(2\omega_j) \\ \hat{\delta}_j &:= \delta(p_j^2 - m_{\kappa_j}^2) = \delta_j^+ + \delta_j^-. \end{aligned}$$

$\overline{\alpha}$ denotes the complex conjugate of α. Summation notation is used for generalized functions, $\int dx\, T(x)f(x) := T(f)$.

1.2 Fields

Fields $\Phi(x)_\kappa$ are elements of an algebra, the noncommutative ring of sums and products of the fields and complex numbers. $x \in \mathbf{R}^d$ and $\kappa \in \{1, 2, \dots N_c\}$. The Wightman functions are VEV of products of fields.

$$\begin{aligned} W_n((x)_n)_{\kappa_1\dots\kappa_n} &:= \langle\Omega|\Phi(x_1)_{\kappa_1}\dots\Phi(x_n)_{\kappa_n}\Omega\rangle \\ &= \langle\Phi(x_k)_{\kappa_k}\dots\Phi(x_1)_{\kappa_1}\Omega|\Phi(x_{k+1})_{\kappa_{k+1}}\dots\Phi(x_n)_{\kappa_n}\Omega\rangle \end{aligned} \tag{2}$$

independently of k. The VEV are the components of the Wightman-functional

$$\underline{W} = (1, W_1(x)_{\kappa_1}, \dots W_n((x)_n)_{\kappa_1\dots\kappa_n}, \dots)$$

defined on a Borchers-Ullmann algebra $\mathcal{A}$ of function sequences. A Borchers-Uhlmann algebra consists of terminating sequences [2,3]

$$\underline{f} := (\, f_0, \dots f_n((x)_n)_{\kappa_1\dots\kappa_n}, \dots).$$

Each $f_n((x)_n)_{\kappa_1\dots\kappa_n}$ is one of a sequence of $(N_c)^n$ functions, one for each selection of $\kappa_1, \dots \kappa_n$. f_0 is a complex number. With $(\xi)_{I_n} := (p, \kappa)_{I_n}$, $(-\xi)_{I_n} := (-p, \kappa)_{I_n}$ and

$$\int (d\xi)_{I_n} := \sum_{\kappa_{i_1}=1}^{N_c} \dots \sum_{\kappa_{i_n}=1}^{N_c} \int dp_{i_1} \dots dp_{i_n}$$

for any set of indices $I_n = \{i_1, i_2, \dots i_n\}$, $\underline{W}$ provides a sesquilinear function on $\mathcal{A} \times \mathcal{A}$.

$$\begin{aligned} \langle\underline{f}|\underline{g}\rangle &= \sum_n \int (d\xi)_n\, \tilde{W}_n((\xi)_n)\sum_{\ell=0}^n \tilde{f}_\ell^*((\xi)_\ell)\, \tilde{g}_{n-\ell}((\xi)_{\ell+1,n}) \\ &= \underline{W}(\underline{f}^* \mathbf{x}\, \underline{g}) \end{aligned} \tag{3}$$

with the product

$$\underline{f}\,\mathbf{x}\,\underline{g} := (f_0 g_0, \dots, \sum_{\ell=0}^n f_\ell((x)_\ell)_{\kappa_1\dots\kappa_\ell}\, g_{n-\ell}((x)_{\ell+1,n})_{\kappa_{\ell+1}\dots\kappa_n}, \dots), \tag{4}$$

and the dual

$$\begin{aligned} \tilde{f}_n^*((\xi)_n) &:= ((D^T\cdot)_n \overline{\tilde{f}_n}((-\xi)_{n,1})) \\ &= \sum_{\ell_1} \dots \sum_{\ell_n} D_{\ell_1\kappa_1}\dots D_{\ell_n\kappa_n} \overline{\tilde{f}_n}(-p_n, -p_{n-1}, \dots, -p_1)_{\ell_n\dots\ell_1}. \end{aligned} \tag{5}$$

Dirac conjugation D is selected so that the dual satisfies $\underline{f}^{**} = \underline{f}$, $(\lambda \underline{f})^* = \overline{\lambda} \underline{f}^*$, $(\underline{g} + \underline{f})^* = \underline{g}^* + \underline{f}^*$ and $(\underline{g} \mathbf{x} \underline{f})^* = \underline{f}^* \mathbf{x} \underline{g}^*$. This dual is an involution [4] when it is an automorphism. This dual (5) is an automorphism of $\mathcal{A}$ but not of $\mathcal{B}$.

(2) and (3) associate the field with multiplication of function sequences in $\mathcal{A}$,

$$\Phi(f)_\kappa \, \underline{g} = \underline{f} \mathbf{x} \underline{g}$$

when $\underline{f} = (0, \ldots 0, f_\kappa(x), 0 \ldots)$. Should there be a field operator in the constructed Hilbert space, it is not Hermitian for the non-trivial constructions since $f = f^* \in \mathcal{B}$ implies that $\|\underline{f}\| = 0$ with $\mathcal{B} \subset \mathcal{A}$ the subalgebra of function sequences used to construct a Hilbert space representation of the quantum mechanics. For the free field Wightman-functional and related physically trivial constructs, the semi-norm provided for $\mathcal{B}$ by (3) extends to $\mathcal{A}$ and consequently the field is Hermitian. The Jost-Schroer theorem provides that no such extension is available for the non-trivial constructions of [1].

1.3 Algebras of test function sequences

A QFT is a local, Poincaré covariant, positive energy supported semi-norm on a subalgebra $\mathcal{B}$ of a Borchers-Uhlmann algebra $\mathcal{A}$. In [2], the algebra of Schwartz functions was denoted Σ and for the constructions, a distinct algebra is designated $\mathcal{A}$. $\mathcal{A}$ has component energy-momentum functions $\tilde{f}_n((p)_n)_{\kappa_1 \ldots \kappa_n}$ that are the multiple argument versions of the span of products of a tempered test function $\tilde{f}(\mathbf{p}) \in S(\mathbf{R}^{d-1})$ and a multiplier $\tilde{g}(p)$ of test functions from $S(\mathbf{R}^d)$ [5].

$$\tilde{f}_1(p) := \tilde{g}(p)\tilde{f}(\mathbf{p}) \in \mathcal{A}.$$

To include massless particles, $\tilde{g}(p)$ and all its derivatives are required to vanish at $E = 0$. The points with $E_j = 0$ are of no consequence in the case of finite masses since the VEV have no support for vanishing energy. With the inclusion of massless particles, this constraint excludes the points where positive and negative mass shells coincide. The component functions of $\mathcal{A}$ are test functions of $(\mathbf{p})_n$ when evaluated on mass shells, $\tilde{f}_n((\pm\omega, \mathbf{p})_n)_{\kappa_1 \ldots \kappa_n} \in S(\mathbf{R}^{n(d-1)})$. The constructed generalized functions [1] have the form

$$\int dE \, d\mathbf{p} \, \delta(E \pm \omega) T(\mathbf{p}) \tilde{f}_1(p) = \int d\mathbf{p} \, T(\mathbf{p}) \tilde{f}_1(\pm\omega, \mathbf{p})$$

for each of the multiple arguments and components, and $T(\mathbf{p}) \in S'(\mathbf{R}^{d-1})$, a generalized function.

Factors of ω_j are not multipliers of test functions [5] when massless elementary particles are included. $\omega_j = \sqrt{\mathbf{p}_j^2}$ in this case. To accommodate massless particles, the functions in the subalgebra $\mathcal{B}$ are set to vanish for all negative energies while preserving that the functions are test functions on mass shells. $\mathcal{B}$ is a subset of $\mathcal{A}$ consisting of functions that vanish on negative energies. For every $f_n \in \mathcal{A}$, let

$$\tilde{\varphi}[f_n]((p)_n)_{\kappa_1 \ldots \kappa_n} := \prod_{k=1}^{n} h(E_k/\beta_k) \tilde{f}_n((p)_n)_{\kappa_1 \ldots \kappa_n}. \tag{6}$$

with $h(E) = 0$ for all $E \leq 0$ and the infinitely differentiable $h(E)$ and all of its derivative vanish at $E = 0$. $\beta_k > 0$. For example,

$$h(E) = \begin{cases} \exp\left(-\frac{1}{E}\right) & E > 0 \\ 0 & E \leq 0. \end{cases} \tag{7}$$

$\varphi[1] := 1$. $\mathcal{B}$ is $\varphi[\mathcal{A}]$ and $\underline{W}(\varphi[\underline{f}])$ is bounded when $\underline{W}(\underline{f})$ is bounded.

LSZ states, used to define scattering amplitudes, appear naturally in the development of [1]. The LSZ states are based upon

$$\tilde{\ell}(p_k) = (\omega_k + E_k)e^{i(\omega_k - E_k)t}\tilde{f}(\mathbf{p}_k).$$

Evaluation of the scattering amplitudes including massless particles for the LSZ states succeeds for the particular VEV studied below. Scattering amplitudes can be evaluated for LSZ states even though the LSZ states are not elements of $\mathcal{B}$, and the results are as presented in [1] with the inclusion of the massless particle descriptions. On positive energies, the LSZ states are limits as a mass parameter becomes arbitrarily small of elements within $\mathcal{B}$.

2 The VEV

For every free field $\underline{W}_o$, there is a family of non-trivial Wightman-functionals $\underline{W}$ [1]. One or more constituent elementary particles are included in the free field description. The VEV $\underline{W}$ result from generators constructed as symmetrized products of a generator $\mathcal{G}_o$ for the free field Wightman-functional $\underline{W}_o$, generators $\mathcal{G}_{n,m}$ for the higher order truncated functions, and an energy ordering $\Theta_{k,n}$. The VEV are

$$\tilde{W}_n((\xi)_n) := \left(\prod_{j=1}^{n}\frac{\partial}{\partial\alpha_j}\right) \sum_{k=0}^{n} \left(\frac{\mathbf{S}[\mathcal{G}_{k,n-k}((\alpha,\xi)_{n+m})\Theta_{k,n}\mathcal{G}_o((\alpha,\xi)_n)]}{k!\,(n-k)!}\right) \tag{8}$$

evaluated at $(\alpha)_n = 0$. $\Theta_{k,n} = 1$ when $-E_j > 0$ for every $j \leq k$ and $E_j > 0$ for $k < j \leq n$ and $\Theta_{k,n} = 0$ otherwise. This form exhibits a unique vacuum (is indecomposable [2]) and reduces to the VEV of a free field $\underline{W}_o$ on $\mathcal{B}$ when the $\mathcal{G}_{n,m} = 1$ [1].

Summation over signed permutations of arguments is denoted by

$$\mathbf{S}[T_n((x)_n)_{\kappa_1...\kappa_n}] := \sum_{\pi} s_{\kappa_{\pi_1}...\kappa_{\pi_n}} T_n(x_{\pi_1}, x_{\pi_2}, \ldots x_{\pi_n})_{\kappa_{\pi_1}...\kappa_{\pi_n}}. \tag{9}$$

The summation includes all $n!$ permutations of $\{1, 2, \ldots n\}$. The signs $s_{\kappa_{\pi_1}...\kappa_{\pi_n}}$ are determined by transpositions, $s_{...\kappa_j\kappa_{j+1}...} = \sigma_{\kappa_j\kappa_{j+1}}s_{...\kappa_{j+1}\kappa_j...}$ with $\sigma_{\kappa_j\kappa_{j+1}} = -1$ if $\kappa_j, \kappa_{j+1} > N_b$, and $\sigma_{\kappa_j\kappa_{j+1}} = 1$ otherwise when $\kappa_j \neq \kappa_{j+1}$. $s_{...\kappa_j\kappa_{j+1}...} = 0$ when $\kappa_j = \kappa_{j+1} > N_b$ and $|s_{\kappa_{\pi_1}...\kappa_{\pi_n}}| = 1$ when no $\kappa_i = \kappa_j > N_b$ for $i \neq j$. These signs are set to agree with the commutation relations of the free field that apply when $x_i - x_j$ is space-like. The argument of $\mathbf{S}[\cdot]$ indicates a term with positive sign and, together with the transpositions, uniquely determines the signs $s_{\kappa_{\pi_1}...\kappa_{\pi_n}}$.

2.1 Free Fields

The free field generator is defined given a two-point functional[a]

$$\begin{aligned}
\tilde{W}_2(p_1, p_2)_{\kappa_1\kappa_2} &= \delta(p_1 + p_2)\,\delta_2^+\,M_{\kappa_1\kappa_2}(p_2) \\
&= \delta(\mathbf{p}_1 + \mathbf{p}_2)\,\delta_1^-\delta_2^+\,2\,\sqrt{\omega_1\omega_2}\,M_{\kappa_1\kappa_2}(p_2).
\end{aligned} \tag{10}$$

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[a]From the Jost-Schroer and related theorems, a field can not be a local Hermitian Hilbert space operator if the interaction is non-trivial and the two-point function has this form [6,7,8]. For the constructions, the field is not a Hermitian operator for models of physical interest. Free fields are an exception.

The elements of the complex valued matrix $M(p)$ are multinomials in energy-momentum components. The condition

$$DM(p) = C^*(p)C(p) \tag{11}$$

results in the free field semi-norm. D is from (5), is a nonsingular linear transformation and satisfies

$$\overline{D}D = 1. \tag{12}$$

$DM(p)$ is nonnegative and consequently $M(p)^* = DM(p)D^T$. For these constructions, $M(p)$ is assumed to have a direct sum decomposition [9] into components $M_k(p)$,

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

and the components satisfy locality conditions

$$M_k(-p)^T = \pm M_k(p). \tag{13}$$

Both components need not be present. The $N_b \times N_b$ boson component M_1 uses '+' and the fermion component M_2 uses '−'. The convention here is that $\kappa \in \{1, 2, \ldots N_b\}$ are boson field components and $\kappa \in \{N_b + 1, \ldots N_c\}$ are fermion field components. Poincaré covariance is implemented in part by

$$\begin{aligned} S(A)M(p)S(A)^T &= M(\Lambda^{-1}p) \\ \overline{S}(A)D &= DS(A) \end{aligned} \tag{14}$$

with $S(A)$ an N_c-dimensional representation of the universal covering group of the proper orthochronous Lorentz group and $A \in \mathrm{SL}(2)$.

There are QFT for every realization for $M(p)$, D, $S(A)$ that satisfy (11), (12), (13) and (14). Example realizations of $M(p)$, D, $S(A)$ for massless models include scalar fields with $M(p) = D = S(A) = 1$. The examples of a two component scalar field with a conserved charge, spin-1 bosons, and spin-1/2 fermions from [1] apply when $m = 0$. The example of spin-1 bosons has a $d \times d$ $D = 1$, $S(A) = \Lambda(A)$ and

$$M(p)_{jk} = p_{(j)}p_{(k)} \qquad \text{or} \qquad M(p)_{jk} = g_{jk}.$$

g is the Minkowski metric. Additional $M(p)$, D, $S(A)$ result from compositions in Kronecker products and direct sums.

Let

$$\mathcal{L}_{I_{\ell,n}} := \prod_{j \in I_{\ell,n}} \frac{\partial}{\partial \alpha_j}$$

for any set of integers with exactly ℓ members $I_{\ell,n} := \{i_1, i_2, \ldots i_\ell\}$ that are an element of the set of subsets of $\chi_1^n := \{1, 2, \ldots n\}$. Summation over $I_{\ell,n}$ includes each of the binomial number ${}_nC_\ell$ subsets of χ_1^n with ℓ elements as well as a summation over $\ell = 0, 1, \ldots n$. $I'_{\ell,n}$ are the complements of $I_{\ell,n}$ in χ_1^n. Similarly for $\chi_{n+1}^{n+m} := \{n+1, n+2, \ldots n+m\}$, $I_{\ell',m}$ and $I'_{\ell',m}$.

The generator for the free field is defined as the multinomial in $(\alpha)_{n+m}$ with generalized function coefficients that results in

$$\begin{aligned} \mathcal{L}_{I_{\ell,n}}\mathcal{L}_{I_{\ell',m}}\mathcal{G}_o((\alpha, \xi)_{n+m}) &:= \tilde{W}_{o;\ell+\ell'}((\xi)_{I_{\ell,n}}, (\xi)_{I_{\ell',m}}) \\ &:= \begin{cases} \displaystyle\sum_{pairs} s_{\pi_1 \ldots \pi_{2k}} \tilde{W}_2(\xi_{\pi_1}, \xi_{\pi_2}) \ldots \tilde{W}_2(\xi_{\pi_{2k-1}}, \xi_{\pi_{2k}}) & \ell + \ell' = 2k \\ \\ 0 & \ell + \ell' = 2k+1 \end{cases} \end{aligned} \tag{15}$$

when $(\alpha)_{n+m} = 0$. $s_{\pi_1...\pi_{2k}}$ is from (9) with $s_{12...2k} = 1$ and the sum is over all $(2k)!/(2^k k!)$ pairs from χ_1^{2k} without regard to order. The indices of the two-point functionals are in ascending index order, $\pi_j < \pi_k$ when $j < k$.

2.2   Higher order truncated functions

The higher order truncated functions result from

$$\tilde{V}_{\ell,\ell'}((\xi)_{I_{\ell,n}}, (\xi)_{I_{\ell',m}}) := \mathcal{L}_{I_{\ell,n}} \mathcal{L}_{I_{\ell',m}} \, \mathcal{G}_{n,m}((\alpha,\xi)_{n+m}) \tag{16}$$

when $(\alpha)_{n+m} = 0$. These generators $\mathcal{G}_{n,m}((\alpha,\xi)_{n+m})$ are Hadamard functions of the $M(p)_{\kappa_1\kappa_2}$ from (10).

$$\ln(\mathcal{G}_{n,m}((\alpha,\xi)_{n+m})) := \int d\zeta_1 \, \overline{z_n((-\xi)_{n,1}, M^*)} \, z_m((\xi)_{n+1,n+m}, M) \times$$
$$\exp(\int d\zeta_2 \, \overline{w_n((-\xi)_{n,1}, DC)} \, w_m((\xi)_{n+1,n+m}, C)) \tag{17}$$

with $DM = C^*C$ from (11) and

$$z_n((\xi)_{a+1,a+n}, M) := \varsigma_n \prod_{\ell=a+1}^{a+n} (1 + \lambda\alpha_\ell e^{-ip_\ell u}\delta_\ell^+ \frac{\partial}{\partial\rho_\ell}) \times$$
$$\exp(\sum_{a<k<j}^{a+n} \rho_k\rho_j \, U_n(p_k-p_j)M_{\kappa_k\kappa_j}(p_k-p_j)) \tag{18}$$
$$w_n((\xi)_{a+1,a+n}, C) := \sum_{j=a+1}^{a+n} e^{-(j-a)v-p_j(s'+s)}\rho_j C(s)_{\ell\kappa_j}.$$

The generator is evaluated at $(\rho)_{n+m} = 0$. The parameters of z_n include the $(\alpha)_{a+1,a+n}$ and $\zeta_1 := \lambda, u$, and for w_n the parameters include $\zeta_2 := s', s, v, \ell$. The ς_n are complex constants with $\varsigma_1 = 0$ to remove the divergent two-point contribution that would result from extrapolating (17) to $n = m = 1$. $(\xi)_{n,1}$ indicates that the indices are in descending order. The indicated summations are

$$\int d\zeta_1 := \int d\sigma(\lambda) \int du$$
$$\int d\zeta_2 := \int d\mu_u(s') \int d\mu_s(s) \int d\mu_\beta(v) \sum_{\ell=1}^{N_c}.$$

$d\mu_s(s)$ and $d\mu_u(s)$ are nonnegative, Lorentz invariant measures with support only for positive energies. These measures correspond with one-dimensional nonnegative tempered measures $d\mu_1(\lambda)$ [10] as

$$d\mu_s(s) = \left(a\delta(s) + \int d\mu_1(\lambda) \, \delta^+(s^2 - \lambda)\right) ds. \tag{19}$$

$d\sigma(\lambda)$ is a nonnegative measure with finite moments,

$$c_n := \int d\sigma(\lambda) \, \lambda^n.$$

Also

$$B_{\kappa_k\kappa_j}(p) := \int d\mu_s(s) \, M_{\kappa_k\kappa_j}(s) \, e^{-sp}$$
$$\Upsilon(p) := \int d\mu_u(s) \, e^{-sp} \tag{20}$$
$$\beta_j := \int d\mu_\beta(v) \, e^{-jv}.$$

The functions $\Upsilon(p)$ and $U_n(p)$ from (18) are Lorentz scalars and multipliers of test functions. For the finite mass cases, $(-p_i + p_j)^2 \geq (m_{\kappa_1} + m_{\kappa_j})^2 > 0$ and the necessary $p^2 = 0$ singularity of non-trivial $\Upsilon(p)$ is beyond the support of the VEV. For the massless case, a sufficient condition to avoid consideration of singularities of $\Upsilon(p)$ is $\Upsilon(p) = c$, a constant. A distinct $\Upsilon(p)$ can be attributed to each constituent matrix $M(p)$ in direct sum compositions.

This construction satisfies the Wightman axioms for QFT with the revision that the algebra of function sequences underlying the Hilbert space of states need not have the involution (5) and consequently that the fields are not necessarily symmetric operators in the Hilbert space [1]. These constructions are described by: a two-point function (10) that determines the constituent elementary particles; truncated function coefficients c_n that are the moments of a nonnegative measure $d\sigma(\lambda)$; constants ς_n with $\varsigma_1 = 0$; Lorentz invariant functions $U_n(p), \Upsilon(p)$ that are multipliers of tempered functions; coefficients β_j that are Laplace transforms (20) of a nonnegative measure $d\mu_\beta(v)$; and a nonnegative, Lorentz invariant measure $d\mu_s(s)$ (19).

3   Continuity

When masses are zero, the singularities of non-trivial $B(p)$ and $\Upsilon(p)$ at $p^2 = 0$ are no longer excluded from the support of the VEV and the ω_j used in [1] to define $\mathcal{B}$ are no longer multipliers. The redefinition of $\mathcal{A}$ and $\mathcal{B}$ in (6) removes consideration of ω_j as a multiplier for $m_\kappa = 0$ but singularities of $B(p)$ and $\Upsilon(p)$ are not excluded from the support of the VEV. However, just as for generalized free fields, there are many cases with summable singularities and a constant $\Upsilon(p)$ is sufficient to demonstrate existence of QFT with massless particles.

From (8), the VEV are a sum of terms that are products of free field functionals $W_{o;n}((\xi)_n)$ with the contributions of $V_{n,m}((\xi)_{n+m})$. $V_{n,m}((\xi)_{n+m})$ consists of products of the higher order truncated functions. No factors share arguments. (17), (18) and (20) result in higher order truncated functions of a form

$$\begin{aligned}
T_n((\xi)_{I_n}) = (2\pi)^d c_n \, \delta(p_{i_1} + \ldots p_{i_n}) \, \Big(\delta_{i_1}^- \delta_{i_2}^- \overline{M}_{\kappa_{i_1}, \kappa_{i_2}} (p_{i_1} - p_{i_2}) \ldots \times \\
\delta_{i_{k-1}}^- \delta_{i_k}^+ \beta_{1 - i_{k-1} + i_k} B_{\kappa_{i_{k-1}} \kappa_{i_k}} (-p_{i_{k-1}} + p_{i_k}) \ldots \delta_{i_{n-1}}^+ \delta_{i_n}^+ M_{\kappa_{i_{n-1}} \kappa_{i_n}} (p_{i_{n-1}} - p_{i_n}) \Big).
\end{aligned} \tag{21}$$

The indices $I_n = \{i_1, \ldots i_n\}$ are distinct and $n \geq 4$. The terms include factors in any combination of $\delta^- \delta^- \overline{M}$, $\delta^+ \delta^+ M$ and $\delta^- \delta^+ B$ that result in at least two factors of δ^- and two factors of δ^+. T_n vanishes for odd n. $U_n(p), \Upsilon(p)$ and ς_n are multipliers and need not be considered for continuity.

The $M(p)$ are multipliers of tempered test functions. The $M(p)$ presented above and in [1] are multinomials in the components of p. The products of the mass shell and energy-momentum conserving delta functions define generalized functions for $\mathcal{A}$ in four or more dimensions whether the m_κ vanish are not. A singularity of the products of the mass shell and energy-momentum conserving delta functions is summable for $d \geq 4$. Appendix A includes this demonstration for massless particles, and [1] includes the result for finite mass.

Singularities of $B(-p_i + p_j)$, in particular the singularity necessarily at $(p_i - p_j)^2 = 0$ for non-constant $B(p)$, can be summable. From (19) and (20), the elements of $B(p)$ result from summation over a mass parameter of a Pauli-Jordan function evaluated for an imaginary argument. The demonstration of summability results from the observation that

$$\hat{f}(\mathbf{s}, \lambda) := \int dp \, \delta^+(p^2) e^{-ps} f(p) \tag{22}$$

is a bounded function of rapid decline in $\mathbf{s}$ and λ when $s = (\omega_\lambda, \mathbf{s}) \in \bar{V}^+$,

$$\omega_\lambda := \sqrt{\lambda + \mathbf{s}^2}$$

for $\lambda \geq 0$ and $f(p) \in \mathcal{A}$. Two momenta define a plane with an angle θ between the two momenta $\mathbf{p}_i$ and $\mathbf{p}_j$. From (21), the energies within a factor of $B(-p_i + p_j)$ are both nonnegative, $-E_i \geq 0$ and $E_j \geq 0$. For massless particles, there is a reference frame with $-p_i + p_j = (\omega_i + \omega_j, \omega_j - \omega_i \cos\theta, -\omega_i \sin\theta, 0\ldots)$ in more than two dimensions. For $s = (\omega_\lambda, \mathbf{s})$, the Cauchy-Schwarz inequality provides that

$$\begin{aligned}
(-p_i + p_j)\, s \;&= (\omega_i + \omega_j)\,\omega_\lambda - (\omega_j - \omega_i \cos\theta)\, s_{(1)} + \omega_i \sin\theta\, s_{(2)} \\
&\geq (\omega_i + \omega_j)\,\omega_\lambda - \sqrt{\omega_j^2 + \omega_i^2 - 2\omega_j\omega_i\cos\theta}\,\sqrt{\mathbf{s}^2} \\
&\geq 0
\end{aligned}$$

since $\omega_i + \omega_j \geq \sqrt{\omega_j^2 + \omega_i^2 - 2\omega_j\omega_i\cos\theta}$ and $\omega_\lambda \geq \sqrt{\mathbf{s}^2}$. This implies that both $-sp_i \geq 0$ and $sp_j \geq 0$, and consequently that $|e^{-sp}| \leq 1$ in (22). Then the summations

$$\int dp\, \delta^+(p^2)\, e^{-ps}\, \left(\omega\tfrac{d}{d\mathbf{p}}\right)^\ell f(p) = \tfrac{1}{2}\int d\mathbf{p}\, e^{-\omega\omega_\lambda + \mathbf{p}\cdot\mathbf{s}}\, \tfrac{d}{d\mathbf{p}}\left(\omega\tfrac{d}{d\mathbf{p}}\right)^{\ell-1} f(\omega, \mathbf{p})$$

$$= \tfrac{1}{2}\int d\mathbf{p}\, (\mathbf{p}\,\omega_\lambda - \omega\,\mathbf{s})e^{-\omega\omega_\lambda + \mathbf{p}\cdot\mathbf{s}}\,\tfrac{d}{d\mathbf{p}}\left(\omega\tfrac{d}{d\mathbf{p}}\right)^{\ell-2} f(\omega, \mathbf{p}),$$

and continuing to integrate by parts until no derivatives remain, are uniformly and absolutely convergent since $f(\omega, \mathbf{p})$ is a test function [11]. From the dominance of terms, it follows that $\hat{f}(\mathbf{s}, \lambda)$ exhibits rapid decline at large values of λ and $\mathbf{s}$. Then the tempered growth countable norms are satisfied providing that $\hat{f}(\mathbf{s}, \lambda)$ is a bounded function of rapid decline when $f(\omega, \mathbf{p}) \in \mathcal{A}$. $\hat{f}(\mathbf{s}, \lambda)$ is infinitely differentiable except at the origin $\mathbf{s} = \lambda = 0$.

For $f(p), g(p) \in \mathcal{A}$, the summation of (20) converges.

$$\begin{aligned}
\int dp_i dp_j\; &B(-p_i + p_j)\delta_i^- \delta_j^+ f(p_i)g(p_j) \\
&= \int d\mu_1(\lambda) \int ds\, \delta^+(s^2 - \lambda)\, M(s) \int dp_i dp_j\, \delta_i^- \delta_j^+ e^{(p_i - p_j)\, s} f(p_i)g(p_j) \qquad (23) \\
&= \int d\mu_1(\lambda) \int ds\, \delta^+(s^2 - \lambda)\, M(s)\, \hat{f}(\mathbf{s}, \lambda)\hat{g}(\mathbf{s}, \lambda).
\end{aligned}$$

$M(s)$ is a multiplier and the summation over λ converges when $d\mu_1(\lambda)$ is of tempered growth, similar to the result in a Källén-Lehmann representation of a two-point function.

Since the constructed factors share no arguments, demonstration that each factor is a generalized function is sufficient to define the product. A demonstration that (21) defines a continuous linear functional suffices given that the free field is well-defined. Factors (21) in $\underline{W}$ with no factors of $B(p)$ are continuous linear functionals, the result of Appendix A since the $M(p)$ are multipliers. When there are one or more factors of $B(p)$ in (21), (23) suffices to define a generalized function. Absorbing the multipliers $M(p)$ into the test function $f((p)_n)$, the p_j can be relabeled to achieve the form

$$\begin{aligned}
\hat{f}((\mathbf{s}, \lambda)_k, (p)_{2k+1,n}) \;&:= \int d(p)_{2k}\, \delta(p_1 + \ldots p_n)\prod_{\ell=1}^{k} e^{(p_\ell - p_{\ell+k})s_\ell}\, \delta_\ell^- \delta_{\ell+k}^+ f((p)_n) \\
&= \int_{\mathcal{M}} d(p)_{2k-1} \prod_{\ell=1}^{k} e^{(p_\ell - p_{\ell+k})s_\ell}\, \delta_\ell^- \delta_{\ell+k}^+ f((p)_n)
\end{aligned}$$

with the latter summation within the manifold $\mathcal{M}$ with $p_k = -p_1 \ldots - p_{k-1} - p_{k+1} \ldots - p_n$ and $2k \leq n$. Since $p_k - p_{2k}$ is linearly independent of the remaining arguments $p_\ell - p_{\ell+k}$ for $\ell = 1, 2, \ldots k-1$ and each $-p_\ell + p_{\ell+k}$ has nonnegative energy, $\hat{f}((\mathbf{s}, \lambda)_k, \ldots)$ is of rapid decline as any $|\mathbf{s}_j|$ or λ_j grows without bound. Appendix A provides that the summation

$$\int d(p)_{n-2k} \, (\delta^\pm)_{2k+1,n} \, \hat{f}((\mathbf{s}, \lambda)_k, (p)_{2k+1,n})$$

converges and exhibits rapid decline in $\mathbf{s}$ and λ. Consequently, the $\mathbf{s}$ and λ summations of (23) converge and the components of $\underline{W}$ are continuous linear functionals.

Then, sufficient conditions to include massless particles in the definition of a Wightman-functional (8) as a continuous linear functional are: the revision to $\mathcal{A}$ and $\mathcal{B}$, $d \geq 4$, $d\mu_1(\lambda)$ is a tempered nonnegative measure.

Appendix

A. The generalized function $\delta(p_1 + \ldots p_n) \, (\delta^-)_k \, (\delta^+)_{k+1,n}$ for massless particles

These generalized functions of $(p)_n \in \mathbf{R}^{nd}$ are weighted summations over the surface with energy and momentum conserved, and energies on mass shells. Here the definition of

$$\delta(p_1 + p_2 + \ldots p_n) \, (\delta^-)_k \, (\delta^+)_{k+1,n}$$

is revisited from [1] for cases including $m_\kappa = 0$. The result is that to include massless particles, $d \geq 4$ for the singularities of the measure on the surface $(\mathbf{p})_n \in \mathbf{R}^{n(d-1)}$ induced by

$$\delta(P_k) := \delta(\omega_1 \ldots + \omega_k - \omega_{k+1} \ldots - \omega_n) \, \delta(\mathbf{p}_1 + \mathbf{p}_2 \ldots + \mathbf{p}_n) \tag{24}$$

to be summable. For massive particles, the generalized function is defined for $d \geq 3$. With $(\mathbf{p})_n$ constrained to the manifold with $\mathbf{P}_k = 0$,

$$\mathbf{p}_n = -\mathbf{p}_1 \ldots - \mathbf{p}_{n-1}, \tag{25}$$

summation over the surface that satisfies $P_{k(0)} = 0$ defines a generalized function except possibly for points on the surface with a vanishing gradient [11].

Define s_j by

$$P_{k(0)} = \sum_{j=1}^{n} s_j \omega_j$$

or $s_i := 1$ for $i \leq k$, and $s_i := -1$ for $i > k$. The cases $k = 0, n$ have no interesting solutions to $P_{k(0)} = 0$ and are not considered below. Then $s_1 = -s_n = 1$. On the indicated manifold, the components of the gradient are

$$\frac{\partial P_{k(0)}}{\partial p_{j(\ell)}} = s_j \frac{p_{j(\ell)}}{\omega_j} - s_n \frac{p_{n(\ell)}}{\omega_n} \tag{26}$$

for $j = 1, \ldots n-1$ and $\ell = 1, \ldots d-1$. The components of $\mathbf{p}_j$ are designated $p_{j(1)}, p_{j(2)}, \ldots p_{j(d-1)}$.

For cases including both nonzero and zero masses, the generalized function (24) is always defined since the gradient (26) never vanishes. When $m_{\kappa_j} = 0$,

$$\mathbf{u}_j := \frac{\mathbf{p}_j}{\omega_j} = \frac{\mathbf{p}_j}{\|\mathbf{p}_j\|}$$

is a unit vector $\|\mathbf{u}_j\| = 1$ and when $m_{\kappa_i} > 0$,

$$\mathbf{v}_i := \frac{\mathbf{p}_i}{\omega_i} = \frac{\mathbf{p}_i}{\sqrt{m_{\kappa_i}^2 + \mathbf{p}_i^2}}$$

are the components of a vector of length strictly less than unity, $\|\mathbf{v}_i\| < 1$. These vectors can never be equal for finite $\mathbf{p}_j, \mathbf{p}_i$ and consequently (26) is never satisfied. Only cases with all $m_\kappa = 0$ or all $m_\kappa > 0$ can be singular. The case with all $m_\kappa > 0$ has a summable singularity for $d \geq 3$ [1]. In models that include both finite and zero masses, truncated functions that include only zero masses or only finite masses occur.

When all $m_\kappa = 0$, the rest mass does not set an energy scale and there is conformal invariance. Whenever $\mathbf{p}_j$ satisfies (26) then $\beta_j \mathbf{p}_j$ is also a solution for any $\beta_j \neq 0$. With $\mathbf{p}_j = \omega_j \mathbf{u}_j$, $\mathbf{u}_j$ a unit vector, the condition for a vanishing gradient (26) becomes $\mathbf{u}_j = s_j \mathbf{u}_1$ and satisfaction of a vanishing gradient is decoupled from energy conservation, $\sum s_j \omega_j = 0$. A neighborhood V of those points in the manifold (25) with a vanishing gradient and satisfying energy conservation is

$$\mathbf{u}_j = s_j \mathbf{u}_1 + \mathbf{e}_j \tag{27}$$

for $j = 2, \ldots n-1$ with $\|\mathbf{e}_j\| < \epsilon$ arbitrarily small and the neighborhoods are constrained by

$$-2s_j \mathbf{u}_1 \cdot \mathbf{e}_j = \mathbf{e}_j^2.$$

This constraint preserves the unit vector lengths. Points with $\mathbf{p}_j = 0$ implying that $\omega_j = 0$ are excluded by the selection of functions in $\mathcal{A}$.

In the neighborhood V of points with a vanishing gradient and conserved energy, (27) provides that

$$\begin{aligned}
\mathbf{p}_n &= -\sum_{j=1}^{n-1} \mathbf{p}_j \\
&= -\sum_{j=1}^{n-1} s_j \omega_j \mathbf{u}_1 - \sum_{j=2}^{n-1} \omega_j \mathbf{e}_j \\
&:= -\omega_n(0) \mathbf{u}_1 - \sum_{j=1}^{n} s_j \omega_j(0)\, \mathbf{u}_1 - \sum_{j=2}^{n-1} \omega_j \mathbf{e}_j.
\end{aligned}$$

In V, energy is conserved when $(\mathbf{e})_{2,n-1} = 0$.

$$\sum_{j=1}^{n} s_j \omega_j(0) = 0,$$

and with

$$\omega_n(0) \mathbf{e}_n := -\sum_{\ell=2}^{n-1} \omega_j \mathbf{e}_\ell, \tag{28}$$

to second order in small quantities

$$
\begin{aligned}
\omega_n((\mathbf{e})_{2,n-1}) \ &= \sqrt{\mathbf{p}_n^2} \\
&= \sqrt{\omega_n(0)^2(-\mathbf{u}_1 + \mathbf{e}_n)^2} \\
&= \omega_n(0)\sqrt{1 - 2\mathbf{u}_1 \cdot \mathbf{e}_n + \mathbf{e}_n^2} \\
&\approx \omega_n(0) + \tfrac{1}{2}\omega_n(0)\mathbf{e}_n^2 + \sum_{j=2}^{n-1}\omega_j\mathbf{u}_1 \cdot \mathbf{e}_j \\
&\approx \omega_n(0) + \tfrac{1}{2}\omega_n(0)\mathbf{e}_n^2 - \tfrac{1}{2}\sum_{j=2}^{n-1}s_j\omega_j\mathbf{e}_j^2.
\end{aligned}
$$

Within V, $\omega_j(\mathbf{e}_j) = \omega_j(0)$ for $j = 1, \ldots n-1$, and

$$
\begin{aligned}
P_{k(0)} \ &\approx -\tfrac{1}{2}\omega_n(0)\mathbf{e}_n^2 + \tfrac{1}{2}\sum_{j=2}^{n-1}s_j\omega_j\mathbf{e}_j^2 \\
&:= R^2 \, \alpha
\end{aligned}
$$

with

$$
R^2 := \sum_{j=2}^{n-1}\mathbf{e}_j^2.
$$

This change to polar coordinates has R as the Euclidean length of $(\mathbf{e})_{2,n-1}$. The $\mathbf{e}_j$ satisfy the constraint to preserve the lengths of $\mathbf{u}_j$, and α is independent of R consistent with the approximations in V. The change in coordinates is from $\mathbf{p}_j$ to $(\mathbf{e}_j, \omega_j)$ for each $j = 2, \ldots n-1$, and neglecting small quantities in V, the Jacobian is unity.

Within V, the generalized function factors.

$$
\begin{aligned}
\delta(P_{k(0)}) \ &= \delta(R^2\alpha) \\
&= \frac{1}{R^2}\,\delta(\alpha) + \frac{1}{\alpha}\,\delta(R^2).
\end{aligned}
$$

Since the generalized function is defined for $R > 0$ and α is independent of R, $\delta(\alpha)$ is defined for $R = 0$. Then, (24) defines a generalized function (without regularization) if R^{-2} is locally summable. $d \geq 4$ suffices since the Jacobian for the polar coordinates for $(\mathbf{e})_{2,n-1}$ contributes $R^{(d-2)(n-2)-1}$. $\mathbf{e}_j$ is only $d-2$ dimensional due to the length preserving constraint. The second term vanishes for $d \geq 4$. Accommodation of massless elementary particles results in the physically satisfying result that $d \geq 4$.

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